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93202Q

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2006
Mathematics with Calculus

9.30 am Friday 24 November 2006
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

A 4-page booklet (S-CALCF) containing mathematical formulae and tables has been centre-stapled in the middle of this booklet. Before commencing, carefully detach the Formulae and Tables Booklet and check that none of its pages is blank.

Answer ALL questions.

Write ALL your answers in the Answer Booklet 93202A.

Show ALL working. Start each question on a new page. Number each question carefully.

Check that this Question Booklet 93202Q has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) On a certain clock, the minute hand is 8 cm long and the hour hand is 6 cm long.

How fast, in cm/min, is the distance between the tips of the hands changing at 9 am?

(b) $\ln(1+x) \approx A + Bx + Cx^2$, for $-1 < x \leq 1$, where A, B, and C are constants.

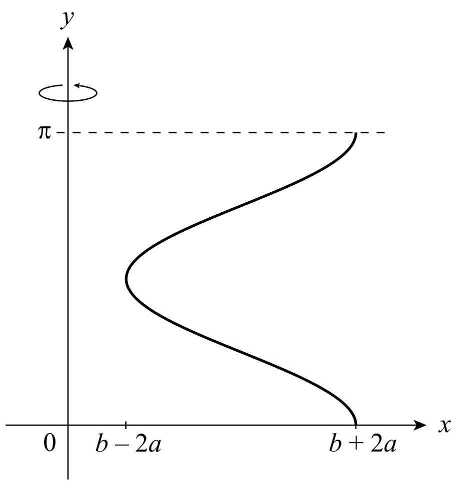
Differentiate twice both sides of the approximation above, and hence show that

$$\ln(1+x) \approx x - \frac{1}{2}x^2.$$

Using logs, estimate the smallest value of the positive integer n for which

$$\left(1 + \frac{1}{2n}\right)^{n+3} < \left(1 + \frac{1}{n}\right)^{n-1}.$$

(c) The diagram below shows the graph of $f(y) = x$ where $x = 2a\cos(2y) + b$, for $a > 0$, $b > 0$ and $0 \leq y \leq \pi$.



By rotating the curve through 2π about the y-axis, we create a model of the shape of the **inner surface** of a vase.

Find an expression in terms of a and b for the volume contained within the inner surface of the vase.

QUESTION TWO

(a) Show that $\cos\frac{\pi}{8}$ can be written exactly as $\frac{\sqrt{2+\sqrt{2}}}{2}$.

Hence, or otherwise, solve the equation $z^2 = \frac{1+7\text{i}}{-3+4\text{i}}$, where z is a complex number and $\text{i} = \sqrt{-1}$, giving exact solutions in the form $a + \text{i}b$, where a and b are constants in surd form.

(b) $f(x) = x^3 - 3x^2 - x + 2$ where $x \leq 1$.

Find an expression for the function $y = g(x)$, where $x \geq 1$, which is obtained by rotating the graph of $y = f(x)$ through 180° about the point $(1, -1)$.

QUESTION THREE

(a) Given the function $y = \sin(\ln x)$, $x > 0$, find the value of k when

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = k.$$

Hence, or otherwise, find a solution of the differential equation

$$\frac{\mathrm{d}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} - y + 5.$$

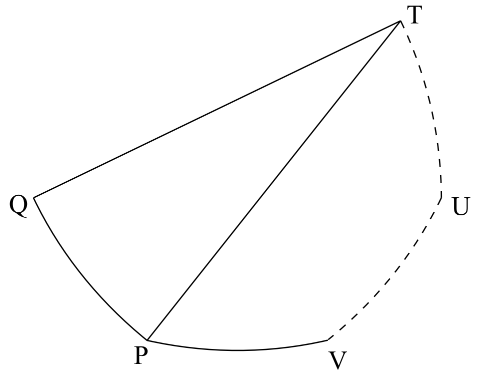
(b) If $\sin(\ln x) = a$ and $\cos(\ln y) = \frac{1}{b}$, $b \neq 0$, and $a \neq b$,

find an expression for $\tan[\ln(xy)].\tan\left[\ln\left(\frac{x}{y}\right)\right]$ in terms of a and b.

Hence find the possible values of a when $\tan[\ln(xy)].\tan\left[\ln\left(\frac{x}{y}\right)\right] = 1$.

QUESTION FOUR

(a) The perimeter of a United Kingdom 50p coin (see below) comprises seven equiangular curves, whose seven vertices, P, Q, R, S, T, U, and V, all lie on a circle, radius r. The curves are obtained by drawing an arc radius R of a circle, centre the vertex opposite the arc, so that, for example, the circular arc PQ has centre T (see diagram below).



Find the ratio of the perimeter of the 50p coin to that of the circumference of the circle its vertices lie on. Your answer should be independent of R and r.

(b) The family of curves $y = f(x)$ has a gradient represented by $y\frac{\mathrm{d}y}{\mathrm{d}x} = 2y + x$.

The curves of this family are perpendicular to a second family $y = g(x)$ at every point of the co-ordinate plane.

Find an implicit equation, in terms of y and x, for this second family of curves.

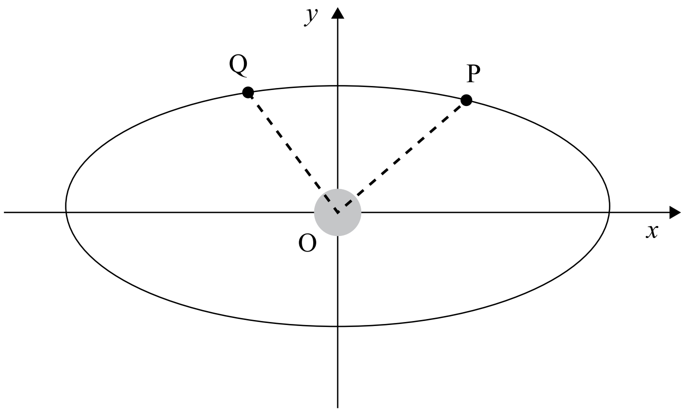
[You may find the substitution $z = \frac{x}{y}$ useful.

You may also use the relationship $\frac{z+2}{z(z+1)} = \frac{2}{z} - \frac{1}{z+1}$.]

QUESTION FIVE

Two satellites, P and Q, are travelling anticlockwise along the same elliptical orbit around a planet centre O, such that the lines OP and OQ are always perpendicular.

The elliptical orbit may be represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, as shown in the diagram below, and the satellite P is represented by the point $(a\cos\theta, b\sin\theta)$.



(a) The satellite P shines a microwave beam in the directions perpendicular to its direction of motion.

Show that, when $\theta = \frac{\pi}{4}$, the beam cuts the vertical plane, represented by the y-axis, at a vertical distance $\frac{\sqrt{2}\,a^2}{2b}$ below the level of the satellite.

(b) Find the co-ordinates of Q in terms of θ.